

Two-Grid Method for Nonlinear Radiation-Diffusion

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What is *this* Two-Grid Method?

- Targeted at nonlinear PDE applications
(fully implicit integration)
- Nonlinear solver enhancement
- Main idea: Nonlinear problem is easier on coarser mesh
- Maybe just a linear solve on fine mesh is enough
- Exchange a fine grid nonlinear problem for a fine grid linear problem
(and a *coarse grid* nonlinear problem)



Two-Grid Background

- Xu ('94,'96) Original analysis for Galerkin FEM (semi-linear and nonlinear equations)
- Dawson & Wheeler ('94): $O(h + H^{3-d/2})$ for lowest order mixed FEM (nonlinear diffusion)
- Wu & Allen ('99): $O(h + H^2)$ for lowest order mixed FEM (semi-linear equations)
- Dawson, Wheeler & Woodward ('98): $O(h^2 + H^{4-d/2})$ for cell-centered finite difference method (nonlinear diffusion)



How Does it Work?

$$\frac{dy}{dt} = f(t, y), \quad B.C. \& I.C.$$

Implicit Integration

CVode: $y_n^p \xrightarrow{NL} y_n$

Two-Grid: $y_n^p \xrightarrow{*} y_n^h \xrightarrow{L} y_n$
 $y_n^{p,H} \xrightarrow{NL} y_n^H$



Implicit Euler Example

- Consider implicit Euler for our IVP:

$$G(\mathbf{y}_n) = \mathbf{y}_n - \mathbf{y}_{n-1} - \delta_n \mathbf{f}(\mathbf{y}_n) = 0$$

- Coarse grid nonlinear problem:

Solve $G(\mathbf{y}_n^H) = 0$ for \mathbf{y}_n^H $\mathbf{y}_n^H \longrightarrow \mathbf{y}_n^h$

- Fine grid linear problem:

Solve $\tilde{G}'(\mathbf{y}_n^h) (\mathbf{y}_n - \mathbf{y}_n^h) = -G(\mathbf{y}_n^h)$ for \mathbf{y}_n

- Use approximate Jacobian: $\tilde{G}'(\xi) = \mathbf{I} - \delta_n \tilde{\mathbf{f}}'(\xi)$.



Why Radiation Diffusion?

- Laser fusion
- Stellar fusion
- Difficult nonlinearities
- Modern applications require ≥ 100 million spatial zones
- Existing implicit solver
- Never been done before



Radiation Diffusion

$$\frac{\partial E_R}{\partial t} = \nabla \cdot \left(\frac{c}{3\rho\kappa_R(T_R)} \nabla E_R \right) + c\rho\kappa_P(T_M) (aT_M^4 - E_R)$$

$$E_R = aT_R^4$$

$$E_M = \text{EOS}(T_M)$$

E_R = radiation energy

T_R = radiation temperature

E_M = material energy

T_M = material temperature

$\kappa_R(T)$ = Rosseland opacity

c = speed of light

$\kappa_P(T)$ = Planck opacity

a = Stefan-Boltzmann const

$\rho(T)$ = material density

t = time



Radiation Diffusion: E_M

$$\frac{\partial E_R}{\partial t} = \nabla \cdot \left(\frac{c}{3\rho\kappa_P(T_R)} \nabla E_R \right) + c\rho\kappa_P(T_M) (aT_M^4 - E_R) + \chi(x)c a T_{\text{source}}^4$$

$$\frac{\partial E_M}{\partial t} = -c\rho\kappa_P(T_M) (aT_M^4 - E_R)$$

$$E_R = aT_R^4 \quad E_M = \text{EOS}(T_M)$$



Discretization

- Cell centered finite difference scheme
- For analysis: frame in terms of mixed FEM
- Mixed FEM methods provide local conservation
- Lowest order methods with specific quadratures reduce to CCFD
- Expanded mixed FEM provides easier formulation on fine grid



Expanded Mixed FEM scheme

Formulation for $-\nabla \cdot (\mathbf{K} \nabla \mathbf{E}) = \mathbf{f}$ gives

$$\tilde{\mathbf{U}} = -\nabla \mathbf{E}, \quad \mathbf{U} = \mathbf{K} \tilde{\mathbf{U}}$$

$$(\nabla \cdot \mathbf{U}, \mathbf{w}) = (\mathbf{f}, \mathbf{w})_{\mathbf{M}}$$

$$(\tilde{\mathbf{U}}, \mathbf{v})_{\mathbf{T}\mathbf{M}} = (\mathbf{E}, \nabla \cdot \mathbf{v})$$

$$(\mathbf{U}, \mathbf{v})_{\mathbf{T}\mathbf{M}} = (\mathbf{K} \tilde{\mathbf{U}}, \mathbf{v})_{\mathbf{T}}$$



How Coarse a Grid?

Nonlinear diffusion and cell centered finite differences $O(h^2 + H^{4-d/2})$ convergence

To maintain $O(h^2)$ convergence:

(For example $h = 1/64,000$)

- 2D: $H = h^{2/3}$ ($H = 1/1600$) (1/40 size)
 - 3D: $H = h^{4/5}$ ($H = 1/7000$) (1/9 size)

Fully Implicit Integration

- Time integration: CVode
(<http://www.llnl.gov/casc/sundials/>)

- Implicit BDF order 1-5
 - Variable order and step-size
 - Solves $\frac{dy}{dt} = f(t, y)$

- Explicit predictor:

$$y_n^p = \sum_{j=1}^q \alpha_{n,j}^p y_{n-j} + h_n \beta_{n,1}^p \dot{y}_{n-1}$$

- Fixed leading coefficient BDF corrector:

$$y_n = a_n + h_n \beta_0 f(t_n, y_n), \quad a_n = \sum_{j=1}^q \alpha_{n,j} y_{n-j} + h_n \beta_{n,1} \dot{y}_{n-1}$$



How to implement this?

- CVode, transport3d, hypre
- Linear interpolation for grid transfers
- Pay attention to nonlinear residual
- Formulate fine grid linear problem
- Theory asserts certain linearizations
- Fudge a little here and there



Interpolation

- Fine grid cell centers \longleftrightarrow coarse grid cell centers (including B.C. effects)
- cell centers \longrightarrow cell corners (due to trapezoid quadratures on diffusion)
- cell interfaces \longrightarrow cell corners (κ_R evaluated at cell interfaces)



Approximate Jacobian

Convergence result requires a particular linearization
of diffusion term:

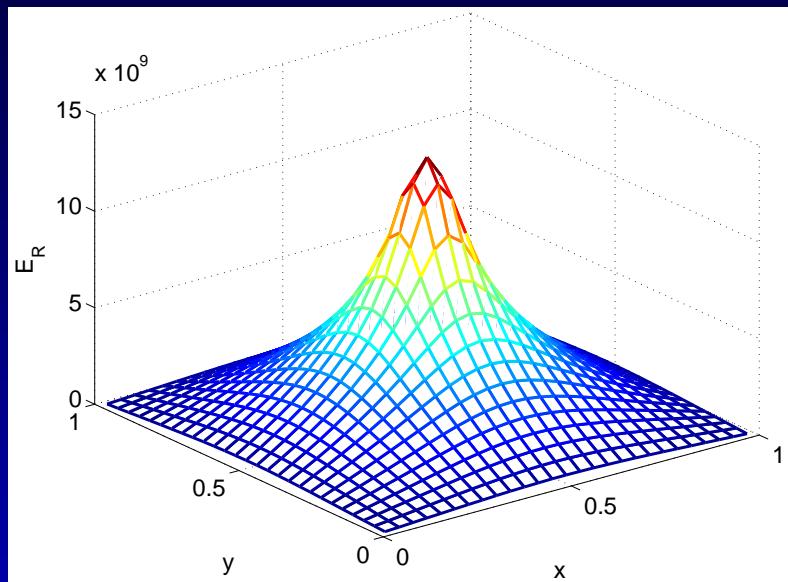
$$\mathbf{K}(\mathbf{E}_{R,h}) \nabla \mathbf{E}_{R,h} \approx \mathbf{K}(\mathbf{E}_{R,H}) \nabla \mathbf{E}_{R,h} + \mathbf{K}'(\mathbf{E}_{R,H}) \nabla \mathbf{E}_{R,H} (\mathbf{E}_{R,h} - \mathbf{E}_{R,H})$$

- Motivated by Taylor expansion
- Coarse cell centers → fine cell corners
- Coarse cell interfaces → fine cell corners
- Next: Newton results for simplified problem

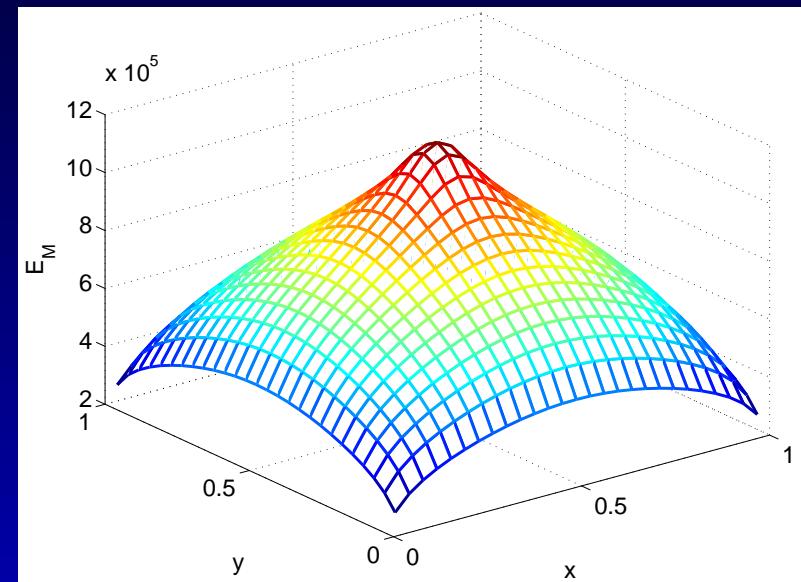


Newton Results

E_R



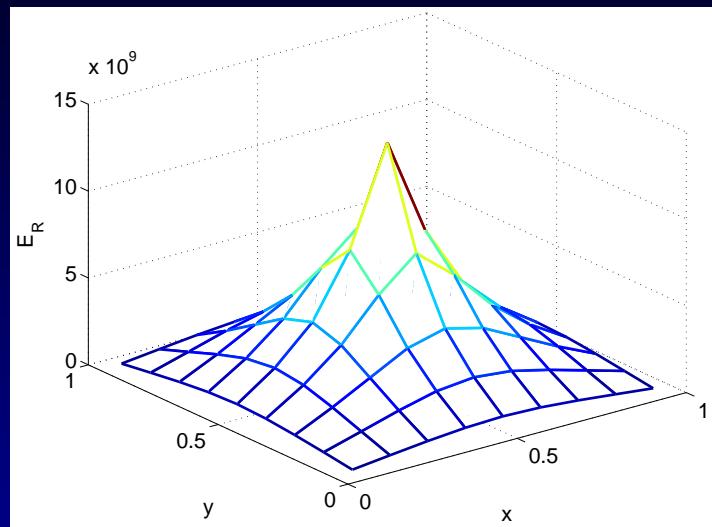
E_M



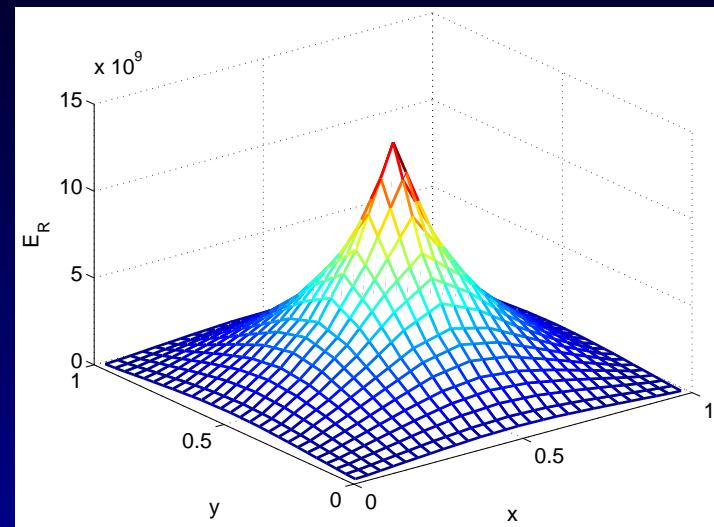
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TwoGrid Results for E_R

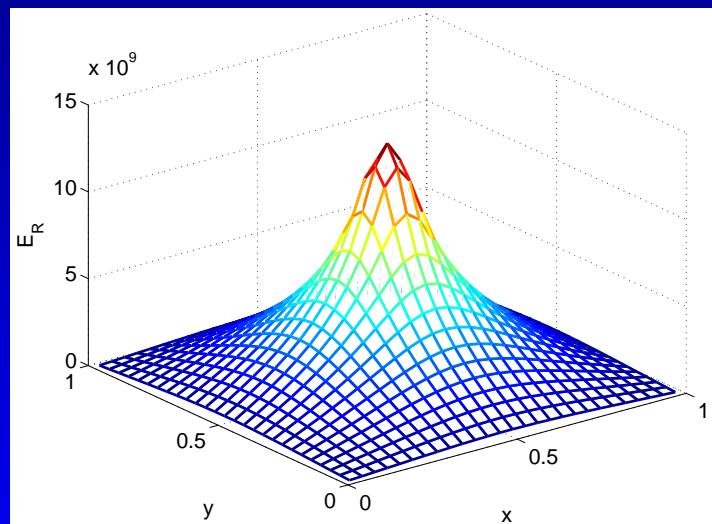
coarse



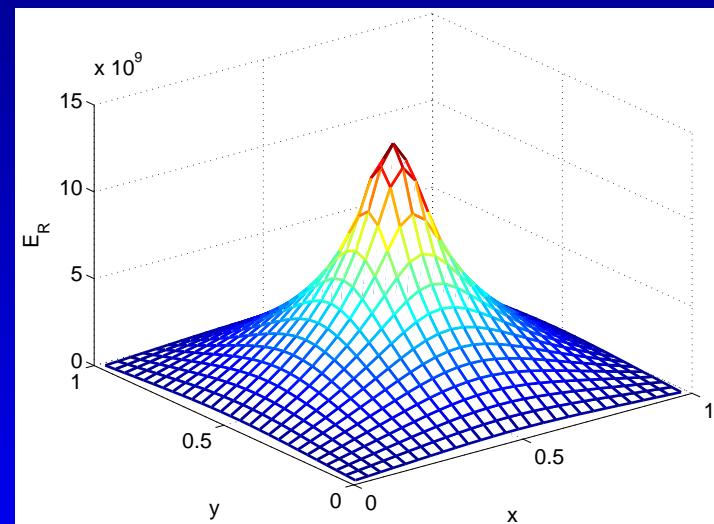
fine



TwoGrid



Newton



What does all this mean?

- TwoGrid method is an appealing nonlinear enhancement
- Implementation is non-trivial
- Initial results are promising
- More work is needed



Where do we go from here?

- Add more nonlinearities
- Verify superconvergence
- Determine whether speedup can be realized
- Consider effect of flux-limiters
- Extend to parallel implementation

Thank you!

